Polynomiography: A New Intersection between Mathematics and Art [1]

Bahman Kalantari

Department of Computer Science, Rutgers University
Hill Center, New Brunswick, NJ, 08903

Polynomiography is defined to be "the art and science of visualization in approximation of the zeros of complex polynomials, via fractal and non-fractal images created using the mathematical convergence properties of iteration functions." An individual image is called a "polynomiograph." The word polynomiography is a combination of the word "polynomial" and the suffix "-graphy." It is meant to convey the idea that it represents a certain graph of polynomials, but not in the usual sense of graphing, say a parabola for a quadratic polynomial. Polynomiographs are obtained using algorithms requiring the manipulation of thousands of pixels on a computer monitor. Depending upon the degree of the underlying polynomial, it is possible to obtain beautiful images on a laptop computer in less time than a TV commercial.

Polynomials form a fundamental class of mathematical objects with diverse applications; they arise in devising algorithms for such mundane task as multiplying two numbers, much faster than the ordinary way we have all learned to do this task (FFT). According to the Fundamental Theorem of Algebra, a polynomial of degree n, with real or complex coefficients, has n zeros (roots) which may or may not be distinct. The task of approximation of the zeros of polynomials is a problem that was known to Sumerians (third millennium B.C.). This problem has been one of the most influential problem in the development of several important areas in mathematics. Polynomiography offers a new approach to solve and view this ancient problem, while making use of new algorithms and today's computer technology. Polynomiography is based on the use of one or an infinite number of iteration functions designed for the purpose of approximation of the roots of polynomials. An iteration function is a mapping of the plane into itself, i.e. given any point in the plane, it is a rule that provides another point in the plane. Newton's iteration function is the best known : $N(z) = z - p(z)/p'(z)$. An iteration function can be viewed as a machine that approximates a zero of a polynomial by an iterative process that takes an input and from it creates an output which in turn becomes a new input to the same machine.

The word "fractal," which partially appears in the definition of polynomiography, was coined by the world-renowned research scientist Benoit Mandelbrot. It refers to sets or geometric objects that are self-similar and independent of scale. This means there is detail on all levels of magnification. No matter how many times one zooms in, one can still discover new details. It turns out that some fractal images can be obtained via simple iterative schemes leading to sets known as Julia sets and the famous Mandelbrot set. The simplicity of these iterative schemes, which may or may not have any significant purpose in mind, has resulted in the creation of numerous web sites in which amateurs and experts exhibit their fractal images. Many fractal images pertain to the famous Mandelbrot set.

Polynomiography, on the other hand, has a well-defined and focused purpose in mind. It makes use of one or an infinite number of iteration functions for polynomial root-finding.

A polynomiograph may or may not result in a fractal image. This is one of the reasons it was necessary to give a new name to this process. A second reason is because its purpose is the visualization of a polynomial, via approximation of its roots and the way in which the approximation is carried out. Even when a polynomiograph is a fractal image it does not diminish its uniqueness. The assertion that an image is a fractal image is no more profound than asserting that an image is a picture, or a painting. In particular, that assertion does not capture any artistic values of the image. Indeed, even in terms of fractal images polynomiography reveals a vast number of possibilities and degrees of freedom and results in a wider variety of images than typical fractal images.

Polynomiography could become a new art form. Working with polynomiography software is comparable to working with a camera or a musical instrument. Through practice, one can learn to produce the most exquisite and complex patterns. These designs, at their best, are analogous to the most sophisticated human designs. The intricate patterning of Islamic art, the composition of Oriental carpets, or the elegant design of French fabrics come to mind as very similar to the symmetrical, repetitive, and orderly graphic images produced through polynomiography. But polynomiographic designs can also be irregular, asymmetric, and non-recurring, suggesting parallels with the work of artists associated with Abstract Expressionism and Minimalism. Polynomiography could be used in classrooms for the teaching of art or mathematics, from children to college-level students, as well as in both professional and non-professional situations. Its creative possibilities could enhance the professional art curriculum.

The "polynomiographer" can create an infinite variety of designs. This is made possible by employing an infinite variety of iteration functions (which are analogous to the lenses of a camera) to the infinite class of complex polynomials (which are analogous to photographic models). The polynomiographer then may go through the same kind of decision making as the photographer: changing scale, isolating parts of the image, enlarging or reducing, adjusting values and colors until the polynomiograph is resolved into a visually satisfying entity. Like a photographer, a polynomiographer can learn to create images that are esthetically beautiful and individual, with or without the knowledge of mathematics or art. Like an artist and a painter, a polynomiographer can be creative in coloration and composition of images. Like a camera, or a painting brush, a polynomiography software can be made simple enough that even a child could learn to operate it.

Despite the significant role of the root-finding problem in the development of fundamental areas, today it is not considered to be a central problem in pure or computational mathematics. According to a 1997 article in SIAM Review by Victor Pan [28], a leading authority on the computational complexity aspects of the root-finding problem, in practice often one needs to compute roots of polynomials of very moderate degree (10 or 20), except possibly in *computer algebra* which is applied to algebraic optimization and algebraic geometry. In view of the above and since there are already efficient subroutines for computing roots of moderate size polynomials, it would not be surprising that many may view the polynomial root-finding problem as one that has basically reached a dead-end. However, I believe that polynomiography will change all of this, not only from the mathematical or scientific point of view, but from the educational and artistic point of view.

Quoting the great American mathematician Smale [34], "There is a sense in which an important result in mathematics is never finished." The Fundamental Theorem of Algebra is one of those results. To me polynomiography is a good evidence in support of Smale's statement. With the availability of a good polynomiography software, the user (who may be a high school student, an artist, or a scientist) is quite likely to wish to experiment with much larger degree polynomials than degree 10 or 20. In particular, this is true because in polynomiography there is a “reverse root-finding problem”: given a polynomial whose roots form a known set of points, find an iteration method whose corresponding polynomiograph would take a desired pattern. In principle, using this reverse problem, I can conceive of the blue print to some of the most elegant patterns, for instance carpets designs, yet to be woven. Their complexity and beauty could very well increase by increasing the degree size. Thus, it is conceivable that the implementation of such designs would demand working with much more computer power than that offered by a laptop computer, perhaps a supercomputer, or a network of computers.

Mathematical Foundation of Polynomiography

Consider the polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $n \geq 2$, and the coefficients $a_0, \ldots, a_n$ are complex numbers. The problem of approximating the roots of $p(z)$ is a fundamental and classic problem. A large bibliography can be found in [25]. For an article on some history, applications, and new algorithms, see Pan [28]. Some of the root-finding methods first obtain a high-precision approximation to a root, then approximate other roots after deflation, see e.g., Jenkins and Traub [10]. Many such root-finding methods make use of iteration functions, e.g. Newton’s or Laguerre’s. A method that guarantees convergence to all the roots was given by Weyl [38]. This method is a two-dimensional analogue of the bisection method. Modifications of this method has been used to obtain an initial approximation to a root, followed by the use of Newton’s method with a guaranteed estimate on the complexity of approximation, see e.g. Renegar [31], Pan [27]. Another root-finding method is based on recursive factorization of the given polynomial, see e.g. Kirrin [23]. Many deep theoretical complexity results on polynomial root-finding and/or the use of Newton’s method, are known, see e.g. Smale [34, 35], Shub and Smale [32, 33], Friedman [6]. Many topics on polynomials can be found in [1], and [2].

One of the conceptually easiest algorithms for the approximation of all the roots of polynomials is described in Kalantari [20], making use of a fundamental family of iteration functions called the “Basic Family.” The algorithm reveals a magical pointwise convergence of the family to roots (see Theorem 2). The Basic Family is represented as $\{B_m(z)\}_{m=2}^{\infty}$. The algebraic development and some optimal properties of the Basic Family are studied in [12], [13]. The first member of the sequence, $B_2(z)$, is Newton’s iteration function, and $B_3(z)$ is Halley’s iteration function [9]. For the rich history of these two iteration functions alone see Ypma [39] and Traub [36]. In particular, Halley’s method inspired the celebrated Taylor’s Theorem (see [39]). Many results on the properties of the members of this family including their close ties with a determinantal generalization of Taylor’s Theorem can be found in [12]-[22].

The members of the Basic Family have an interesting closed formula. Let $p(z)$ be a polynomial of degree $n \geq 2$ with complex coefficients. Set $D_0(z) \equiv 1$, and for each natural number $m \geq 1$, define

$$D_m(z) = \det \begin{pmatrix} p'(z) & \frac{p''(z)}{2!} & \ldots & \frac{p^{(m-1)}(z)}{(m-1)!} & \frac{p^{(m)}(z)}{(m)!} \\ p(z) & p'(z) & \ddots & \ddots & \frac{p^{(m-1)}(z)}{(m-1)!} \\ 0 & p(z) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{p''(z)}{2!} \\ 0 & 0 & \ldots & p(z) & p'(z) \end{pmatrix},$$

$$\widehat{D}_{m,i}(z) = \det \begin{pmatrix} \frac{p''(z)}{2!} & \frac{p'''(z)}{3!} & \dots & \frac{p^{(m)}(z)}{(m)!} & \frac{p^{(i)}(z)}{i!} \\ p'(z) & \frac{p''(z)}{2!} & \ddots & \frac{p^{(m-1)}(z)}{(m-1)!} & \frac{p^{(i-1)}(z)}{(i-1)!} \\ p(z) & p'(z) & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \frac{p''(z)}{2!} & \frac{p^{(i-m+2)}(z)}{(i-m+2)!} \\ 0 & 0 & \dots & p'(z) & \frac{p^{(i-m+1)}(z)}{(i-m+1)!} \end{pmatrix},$$

where $i = m+1, \dots, n+m-1$, and $\det(\cdot)$ represents determinant.

For each $m \geq 2$, define

$$B_m(z) \equiv z - p(z) \;\; \frac{D_{m-2}(z)}{D_{m-1}(z)}.$$

Note that $D_m(z)$ corresponds to the determinant of a Toeplitz matrix defined with respect to the normalized derivatives of $p(z)$. A square matrix is called Toeplitz if its elements are identical along each diagonal.

Some of the fundamental properties of the members of the Basic Family are described in the following theorem. Its second part, in particular is a very special case of a nontrivial determinantal generalization of Taylor's theorem given in [15].

Let $\mathbf{C}$ be the field of complex numbers. For a complex number $c = a + ib$, where $i = \sqrt{-1}$, its modulus is $|c| = \sqrt{a^2 + b^2}$.

Theorem 1. Kalantari [16, 20]) The following conditions hold:

1. For all $m \geq 1$ we have,

$$D_m(z) = \sum_{i=1}^{n} (-1)^{i-1} \frac{p^{i-1}(z) p^{(i)}(z)}{i!} D_{m-i}(z), \quad D_j = 0, \quad j < 0.$$

2. Let θ be a simple root of $p(z)$. Then,

$$B_m(z) = \theta + \sum_{i=m}^{m+n-2} (-1)^m \frac{\widehat{D}_{m-1,i}(z)}{D_{m-1}(z)} (z - \theta)^i.$$

3. There exists $r > 0$ such that given any $a_0 \in N_r(\theta) = \{z : |z - \theta| \leq r\}$, the fixed-point iteration $a_{k+1} = B_m(a_k)$ is well-defined, and it converges to θ having order m. Specifically,

$$\lim_{k \to \infty} \frac{(\theta - a_{k+1})}{(\theta - a_k)^m} = (-1)^{m-1} \frac{\widehat{D}_{m-1,m}(\theta)}{p'(\theta)^{m-1}}.$$

4. Let θ be a simple root of $p(z)$. There exists a neighborhood of θ, $N^*(\theta)$, such that for each a within this neighborhood $p'(a) \neq 0$, $|a - \theta| < 1$, and

$$|p'(a)| - \sum_{i=0, i \neq 1}^{n} |p(a)|^{\frac{i-1}{2}} \frac{|p^{(i)}(a)|}{i!} \geq \frac{1}{2} |p'(a)|.$$

For any $a \in N^*(\theta)$, if we set

$$h(a) = \left[\sum_{i=0}^{n} \left(\frac{|p^{(i)}(a)|}{i!} \right)^2 \right]^{1/2},$$

then we have

$$|B_m(a) - \theta| \leq \left|\frac{2h(a)}{p'(a)}(a-\theta)\right|^m \left(\frac{1}{1-|a-\theta|}\right).$$

5. In particular, if θ is a simple root of $p(z)$, there exists $r^* \in (0,1)$ such that given any $a \in N_{r^*}(\theta)$, we have

$$\theta = \lim_{m\to\infty} B_m(a).$$

6. $D_m(a)$ and hence $B_m(a)$ can be computed in $O(n \log n \log m)$ operations.

To describe a fundamental global convergence property of the sequence let

$$R_p = \{\theta_1, \ldots, \theta_t\}$$

be the set of distinct roots of $p(z)$. The elements of R_p partition the Euclidean plane into Voronoi regions and their boundaries. The Voronoi region of a root θ is a convex polygon defined by the locus of points which are closer to this root than to any other root. More precisely, the Voronoi region of a root θ is

$$V(\theta) = \{z \in \mathbf{C} : |z-\theta| < |z-\theta'|, \ \ \forall \ \ \theta' \in R_p, \ \ \theta' \neq \theta\}.$$

Let S_p be the locus of points that are equidistant from two distinct roots, i.e.

$$S_p = \{z \in \mathbf{C} : |z-\theta| = |z-\theta'|, \ \ \text{where} \ \ \theta, \theta' \in R_p, \ \ \theta \neq \theta'\}.$$

This is a set of measure zero consisting of the union of a finite number of lines.

Definition 1. Given $a \in \mathbf{C}$ the *Basic Sequence* at a is defined as

$$B_m(a) = a - p(a)\frac{D_{m-2}(a)}{D_{m-1}(a)}, \quad m = 2, 3, \ldots.$$

Theorem 2. ([20]) Given $p(z)$, for any input $a \notin S_p$, the Basic Sequence is well-defined satisfying

$$\lim_{m\to\infty} B_m(a) = \theta,$$

for some $\theta \in R_p$. Under some regularity assumptions, e.g. simplicity of all the roots of $p(z)$, for all $a \in V(\theta)$, $\lim_{m\to\infty} B_m(a) = \theta$.

Remark. The determinantal generalization of Taylor's theorem given in part 2 of Theorem 1 gives a mechanism for estimating the error $|B_m(a) - \theta|$. The Basic Sequence can be even defined for functions that are not polynomial. In Kalantari [19] the pointwise evaluation of the Basic Family (the Basic Sequence in the terminology of Definition 1) is used to give new formulas for the approximation of π and e. In Kalantari [20] it is also shown how the relationship between the Basic Sequence and the Basic Family gives new results for general homogeneous linear recurrence relations that are defined via a single nonzero initial condition.

Basins of Attraction and Voronoi Regions of Polynomial Roots

Consider a polynomial $p(z)$ and a fixed natural number $m \geq 2$. The basins of attraction of a root of $p(z)$ with respect to the iteration function $B_m(z)$ are regions in the complex plane such that given an initial point a_0 within them, the corresponding sequence $a_{k+1} = B_m(a_k)$, $k = 0, 1, \ldots,$ will converge to that root. It turns out that the boundary of the basins of attractions of any of the polynomial roots is the same set. This boundary is known as the Julia set and its complement it known as the Fatou set. The fractal nature of Julia sets and the images of the basins of attraction of Newton's method are now quite familiar for some special polynomials. For example fractal image of $p(z) = z^3 - 1$ was apparently first studied by the American mathematician John Hubbard while teaching calculus in France (see Gleick [8]). That image and similar images that are now quite familiar also appear in [26], as well as [37] (see also its references). Mandelbrot's work (see *The Fractal Geometry of Nature* [24]) in particular popularized the Julia theory [11] on the iteration of rational complex functions, as well as the work of Fatou [5], and led to the famous set that bears Mandelbrot's name. Peitgen et al. [30] undertake a further analysis of fractals. Mathematical analysis of complex iterations may be found in Peitgen and Richter [29], Devaney [3], and Falconer [4].

While the fractal nature of the Julia sets corresponding to the individual members of the Basic Family follows from the Julia theory on rational iteration functions, that theory does not predict the total behavior of specific iteration functions on the complex plane. For instance the Julia theory does not even predict the shapes of basins of attraction of the polynomial $p(z) = z^n - 1$ for various members of the Basic Family. In contrast, there are important consequences of the results stated in Theorems 1 and 2 which apply to arbitrary polynomials. In particular, Theorem 2 implies the following: Except possibly for the locus of points equidistant to two distinct roots, given any input a, the Basic Sequence $\{B_m(a) = a - p(a)D_{m-2}(a)/D_{m-1}(a)\}$ converges to a root of $p(z)$. Under some regularity assumption (e.g. simplicity of the roots), for almost all inputs within the Voronoi polygon of a root, the corresponding Basic Sequence converges to that root. The Basic Sequence corresponds to the pointwise evaluation of the Basic Family. Theorem 2 gives rise to a new set of non-fractal polynomiographs with enormous beauty.

Figure 1 and Figure 2 present several fractal images that confirm the theoretical convergence results: as m increases, the basins of attraction of the roots, as computed with respect to the iteration function $B_m(z)$, rapidly converge to the Voronoi regions of the roots. Thus the regions with chaotic behavior rapidly shrink to the boundaries of the Voronoi regions.

Figure 1 considers a polynomial with a random set of roots, depicted as dots. The figure shows the evolution of the basins of attraction of the roots to the Voronoi regions as m takes the values $2, 4, 10$, and 50. Figure 2 shows the basins of attraction for the polynomials $p(z) = z^4 - 1$, corresponding to different values of m. The roots of $p(z) = z^4 - 1$ are the roots of unity and hence the Voronoi regions are completely symmetric. In these figures in the case of $m = 2$, i.e. Newton's method, the basins of attraction are chaotic. However, these regions rapidly improve by increasing m.

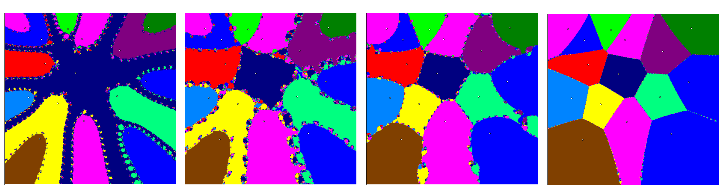

Figure 1: Evolution of basins of attraction to Voronoi regions via $B_m(z)$: random points, $m = 2, 4, 10, 50$ (left to right).

Polynomiography and Visual Arts

I will first describe some general techniques for the creation of polynomiographic images

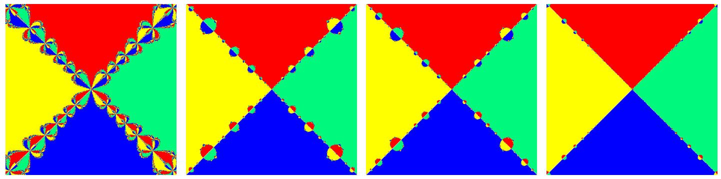

Figure 2: Evolution of basins of attraction to Voronoi regions via $B_m(z)$: $p(z) = z^4 - 1$, $m = 2, 3, 4, 50$ (left to right).

and exhibit sample images obtained via a prototype software for polynomiography, as well as provide an explanation of how some of these images were created. Subsequently, I will consider other applications of polynomiography.

General Techniques for Creating Artwork

In order to describe some general techniques for producing polynomiographic images I will first describe the capabilities of prototype polynomiography software we have developed. The software allows the user to create an image by inputing a polynomial through several means, e.g. by inputing its coefficients, or the location of its zeros. In one approach the user simply inputs a parameter, m, as any natural number greater than one. The assignment of a value for m corresponds to the selection of $B_m(z)$ as the underlying iteration function. This together with the selection of a user-defined rectangular region and user-specified number of pixels, tolerance, as well as a variety of color mapping schemes enables the user to create an infinite number of basic polynomiographs. These polynomiographs turn out to be fractal images and can subsequently be easily re-colored or zoomed in any number of times. A non-fractal and completely different set of images could result from the visualization of the root-finding process through the collective use of the Basic Family, i.e. the pointwise convergence property described in Theorem 2. These images are enormously rich. In either type of image creation technique the user has a great deal of choices, e.g. the ability to re-color any selected regions using a variety of coloration schemes based on convergence properties.

Viewing polynomiography as an art form, one can list at least four general image creation techniques.

(1) Like a photographer who shoots different pictures of a model and uses a variety of lenses, a polynomiographer can produce different images of the same polynomial and make use of a variety of iteration functions and zooming approaches until a desirable image is discovered.

(2) In this more creative approach an initial polynomiograph, possibly very ordinary, is turned into a beautiful image, based on the user's choice of coloration, individual creativity, and imagination. This is analogous to carving a statue out of stone.

(3) The user employs the mathematical properties of the iteration functions, or the underlying polynomial, or both (this is truly a marriage of art and mathematics).

(4) Images can be created as a collage of two or more polynomiographs produced through any one of the previous three methods.

Many other image creation techniques are possible, either through artistic compositional means, or through computer assisted design programs.

An Exhibition

This section presents an exhibition of some polynomiographic images that I have created through the above four techniques. The reader may notice the diversity of these images. In particular, they contrast with literally hundreds of fractal images exhibited at web sites. It should be noted that the images displayed are not necessarily symmetric. These are given as Figure 3 - Figure 13. It should be mentioned that none of the images in the paper is displayed at its optimal size. Indeed the optimal size for the display of some of the images is poster size. This is because at that size we begin to see and appreciate the real complexity and detail of the images.

In the near future an interactive version of a polynomiography software will become available at www.polynomiography.com, where the visitor will be able to input his/her own polynomial and obtain various polynomiographs.

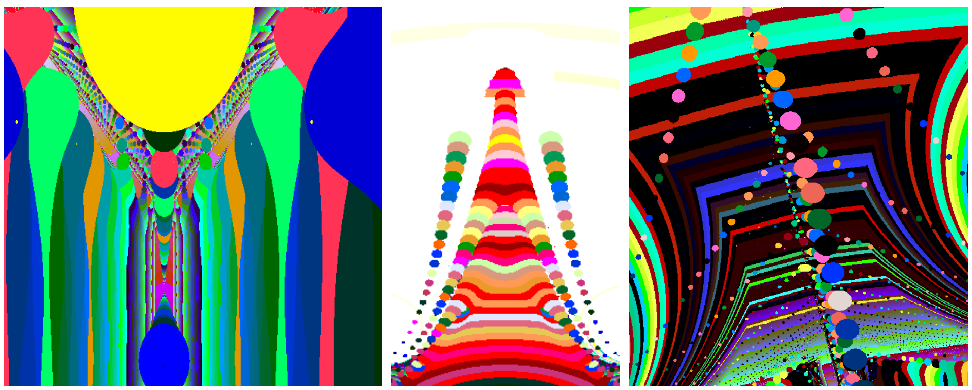

Figure 3: “Cathedral”, “Eiffel Tower”, “Party on Brooklyn Bridge”

Figure 4: “Waltz”

With the exception of some of the images in the “Evolution of Stars and Stripes”, and in

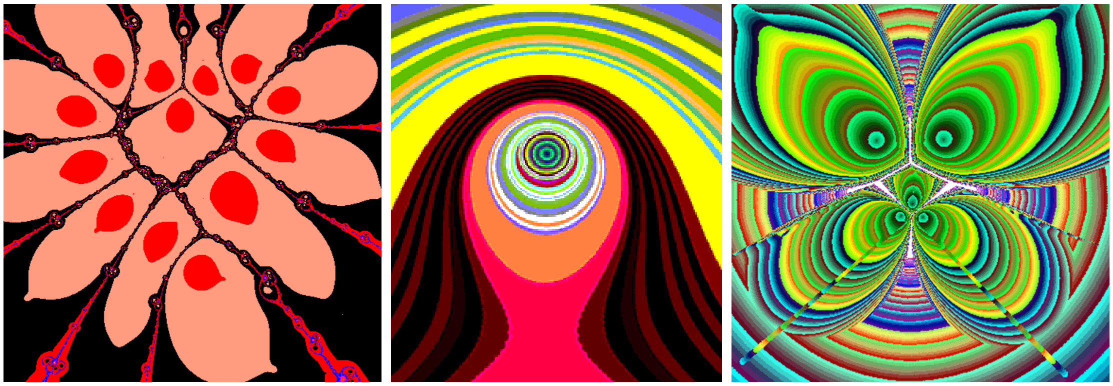

Figure 5: “Mathematics of a Heart”, “Mona Lisa in 2001”, “Butterfly”

Figure 6: All Untitled

“Ms. Poly” all other images have resulted from a single input polynomial. In “Ms. Poly” all regions that depict different features come from a single polynomial, except for her lips, themselves a polynomiograph, which came from a different polynomial and were collaged.

The Making of “Mona Lisa in 2001”

A polynomial of degree 10 was used to produce the left polynomiograph in Figure 12. Then as a result of zooming in on one of the significant parts of that image, the right image in Figure 12 was obtained. A coloration of various layers resulted in the final image.

The Making of “Mathematics of a Heart”

One of the features of the software is to allow the user to enter a polynomial by placing the location of the roots on the working canvas of the computer monitor. The software then builds the polynomial from the roots. Subsequently, any one of the techniques can be applied. The image “Mathematics of a Heart” was produced by simply drawing a heart shape and then by applying one of the techniques, together with personal coloration.

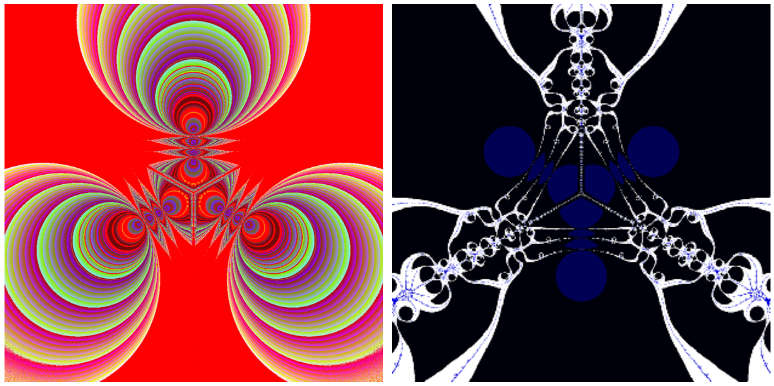

Figure 7: "Life and Death" (two views of the same polynomial)

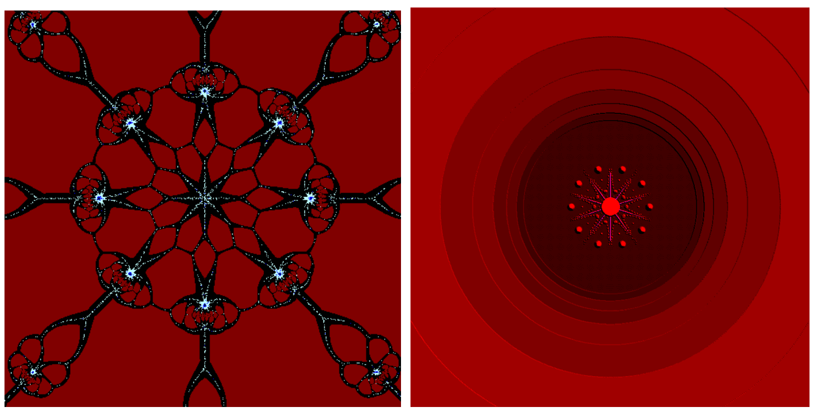

Figure 8: "Acrobats", "Happy Birthday Hal!!!"

The Making of "Evolution of Stars and Stripes"

The inspiration to create a polynomiographic image of the U.S. flag came from Jasper Johns paintings of the flag. The first image was conceived by making use of the convergence properties of the Basic Family and the fact that the basins of attraction converge to the Voronoi regions of the roots. The underlying polynomial for the first image is a polynomial of degree 5 whose roots consist of five points on a vertical line. The second and third images are variations of the first one. The star image was obtained from a single polynomial and is collaged (after reduction of size) onto the third image. The final image comes from the coloration of a polynomiograph of a single polynomial (zoomed in at an appropriate area) together with the collage of white stars coming from a different polynomiograph than that of the evolving star. It is hoped that the five images give the impression that the flag is evolving: the appearance of the stripes and the emergence of the color white (first images); the close up of the first image, while the evolution is in progress, which reveals the emergence of the color blue (second image); the creation of stars and the continuous growth and reshaping of white and blue regions (third image); a close-up of one of the forming stars which appear as if it is in rotation (fourth figure); and finally the orderly formations of various components of the flag and the settling of the stars. The conceptualization and creation of the five images took more than the summer of 2000.

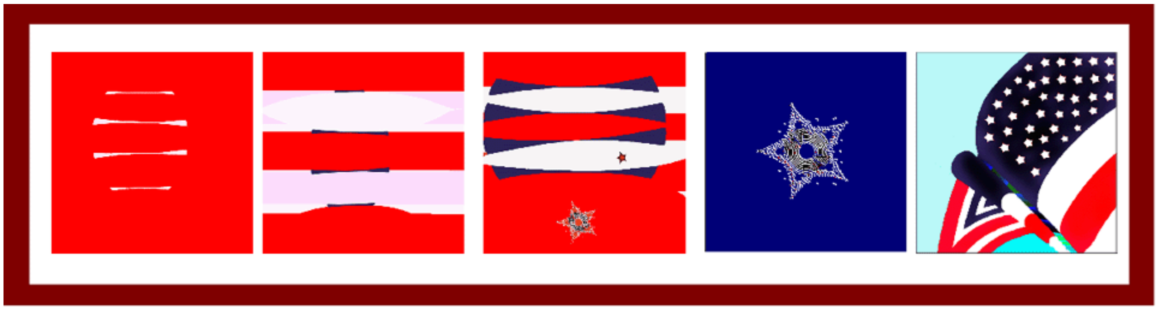

Figure 9: "Evolution of Stars and Stripes"

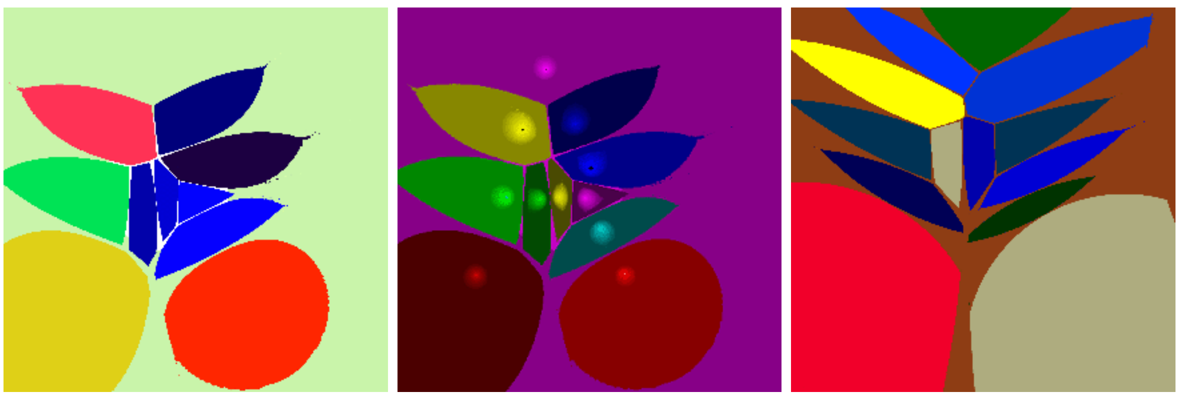

Figure 10: "Summer Variations"

Symmetric Designs from Polynomiography

Symmetric designs can be obtained by considering polynomials whose roots have symmetric patterns. One can obtain interesting images by simply considering polynomials whose roots are the roots of unity or more generally n-th root of a real number r. By the multiplication of these polynomials, as well as the rotation of the roots one can obtain some very interesting basic designs that can subsequently be painted into beautiful designs. The images in Figure 13 and Figure 16 are such examples.

Polynomiographs of Numbers

One interesting application of polynomiography is in the encryption of numbers, e.g. ID numbers or credit card numbers into a two dimensional image that resembles a fingerprint. Different numbers will exhibit different fingerprints. One way to visualize numbers as polynomiographs is to represent them as polynomials. For instance a hypothetical social security number $a_8a_7\cdots a_0$ can be identified with the polynomial $P(z) = a_8z^8 + \cdots + a_1z + a_0$. Now we can apply any of the techniques discussed earlier. A particularly interesting visualization results when the software makes use of the Basic Family collectively (see Theorem 2). Figures

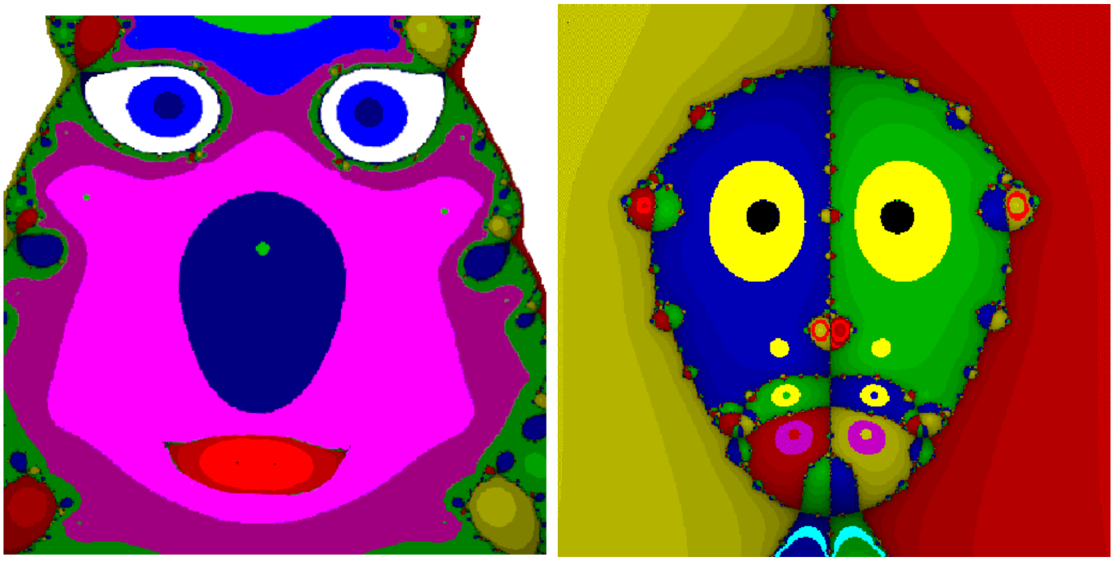

Figure 11: "Ms. Poly" and "L3"

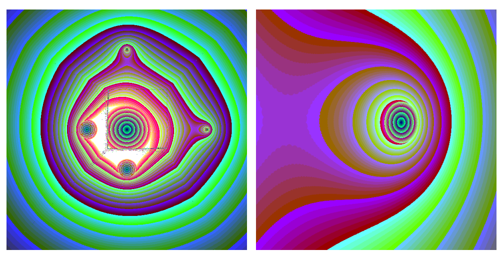

Figure 12: Polynomiographs leading to "Mona Lisa in 2001"

14 gives several examples.

The reader may notice that all of the images are distinct except possibly the two lower rightmost images. But that is because they are consecutive numbers. Upon closer look, Figure 15, their differences can be noticed immediately. Now given such a polynomiograph for a number it should be possible to build scanners that can convert the image back into the original number. The conversion requires the recognition of the roots and the recovery of the corresponding polynomial coefficients.

Public Use of Polynomiography in Arts and Design

Through various software programs, polynomiography could grow into a new art form for both professional and non-professional art-making, and in the teaching of both art and mathematics. The recent proliferation of craft supply stores in the United States reveals the deep seated urge in the general population to create. Having polynomiography software available on home computers could release that creative power even more; people could invent their own knitting and needlepoint patterns, design their own carpets, and create simply for the sake of creating. The polynomiographic pattern in Figure 16 was motivated by a design from a Persian

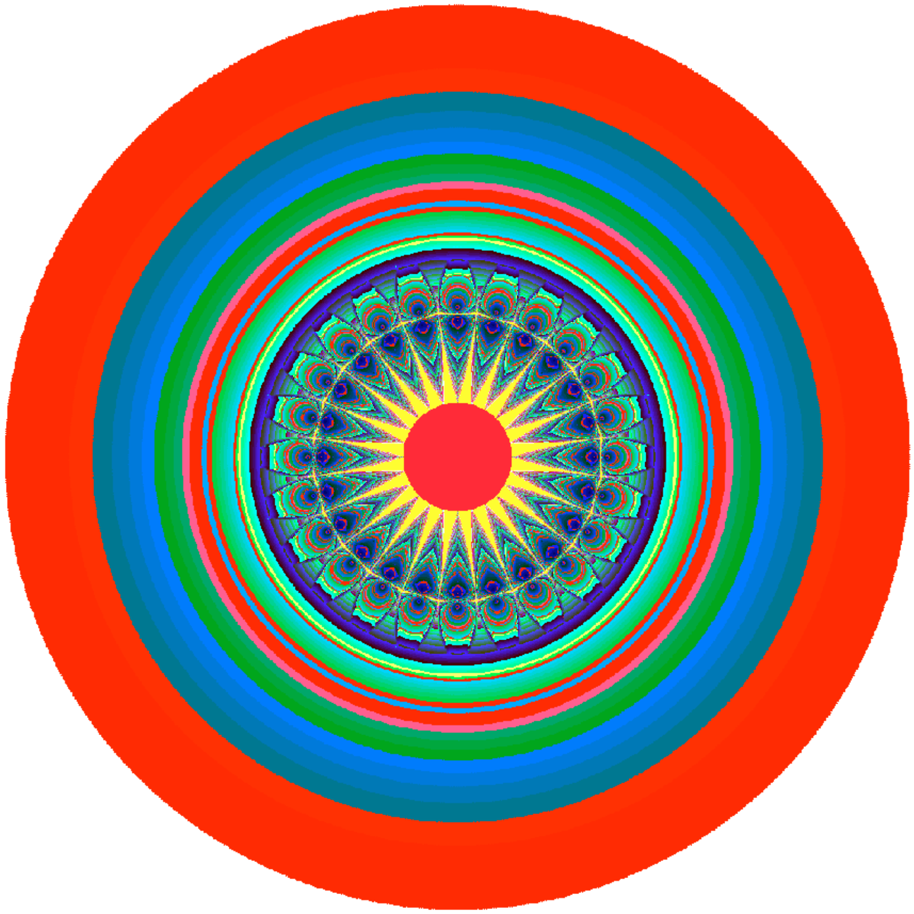

Figure 13: A polynomiograph of $10z^{48} - 11z^{24} + 1$

carpet and made use of pointwise convergence described in Theorem 2. It is possible to obtain much more sophisticated patterns than this.

At a more advanced level, polynomiography can be used to teach the fundamentals of design and color theory. While information science software has transformed the advanced graphic design curriculum through Photo Shop, Quark, and similar programs, the visual arts curriculum makes little use of information science at the fundamentals level. Polynomiography provides the opportunity to teach decision-making in composition, spatial construction, and color at a much more complex level than the present system which relies on the traditional methods of drawing and collage. Through using polynomiography software, students have infinite design possibilities before them. In making their selections, they can experiment with symmetry and asymmetry, repetition, unity, balance, spatial illusion, lines, and planes. By using color, they can gain understanding of complementary color, value relationships, and balance while learning to manipulate hue, tint, and intensity.

Polynomiography and Education

Polynomiography has enormous potential applications in education. A polynomiography

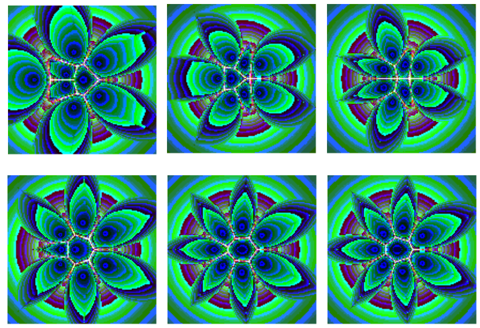

Figure 14: A polynomiograph of six different nine digit numbers

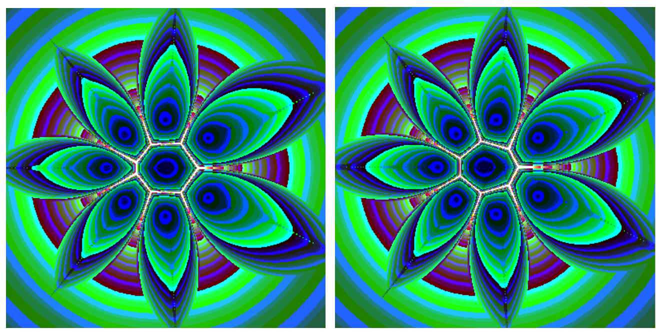

Figure 15: A polynomiograph of the numbers 672,123,450 and 672,123,451

software could be used in the mathematics classroom as a device for understanding polynomials as well as the visualization of theorems pertaining to polynomials. As an example of one application of polynomiography, high school students studying algebra and geometry could be introduced to mathematics through creating designs from polynomials. They would learn to use algorithms on a sophisticated level and to understand the mathematics of polynomials in their relationship to pattern and design in ways that cannot be approached abstractly. For instance, Figure 14 can be viewed as a visualization of the Fundamental Theorem of Algebra, as applied to the polynomials corresponding to these numbers. It is possible to compile many theorems about polynomials and their properties, or those of iteration functions that can be visualizable through polynomiography. At a higher educational level, e.g. calculus or numerical analysis courses, polynomiography allows students to tackle important conceptual issues such as the notion of convergence and limits, as well as the idea of iteration functions, and gives the student the ability to understand and appreciate more modern discoveries such as fractals.

There are also numerous algorithmic issues that are motivated by polynomiography, such as the development of even newer root-finding algorithms. Polynomiography is not only a means for obtaining fast algorithms for polynomial root-finding but also allows the users to determine how two particular members of the Basic Family compare. Which is better, Newton's method or Halley's method? Most numerical analysis books do not bother with such questions. But indeed these are fundamental questions from a pedagogical and practical point of view. One may make use of fractal images to study the computational advantages of members of the Basic Family over Newton's method, as well as the advantages in using the Basic Sequence in computing a single root, or all the roots of a given polynomial.

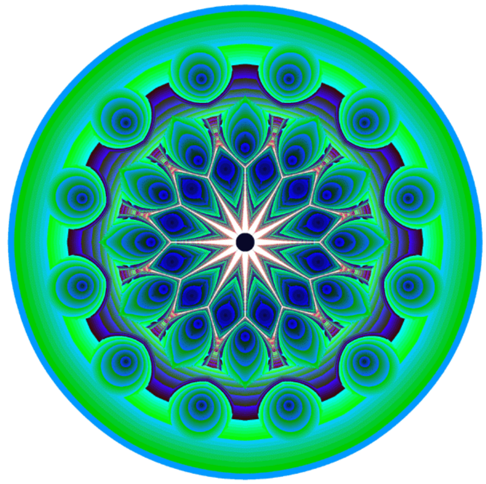

Figure 16: A polynomiograph of a degree 36 polynomial

Extensions of Polynomiography and Iteration Techniques

In this paper, polynomiography has been introduced in terms of the properties of a fundamental family of iteration functions, namely the Basic Family. However, more generally polynomiography can be defined with respect to any iteration function for polynomial root-finding. While experimentation with polynomiographs of other iteration functions is likely to be interesting, many unique mathematical and computational properties of the Basic Family members makes them much more interesting than any other individual iteration function, and as a family more fundamental than any other family of iteration functions. There are still many mathematical properties of the family that need to be studied. Moreover, polynomiography could even be enhanced with multipoint versions of the Basic Family.

A nontrivial determinantal generalization of Taylor's formula proved in Kalantari [15], whose precise description is avoided here, plays a dual role in the approximation of a given function or its inverse (hence its roots). On the one hand, the determinantal Taylor formula unfolds each ordinary Taylor polynomial into an infinite spectrum of rational approximations to the given function. On the other hand, these formulas give an infinite spectrum of rational inverse approximations, as well as single and multipoint iteration functions that include the Basic Family. Given $m \geq 2$, for each $k \leq m$, we obtain a k-point iteration function, defined as the ratio of two determinants that depend on the first $m-k$ derivatives. These matrices are upper Hessenberg and for $k=1$ also reduce to Toeplitz matrices. The corresponding iteration function is denoted by $B_m^{(k)}$. Their formula will not be given here. Their order of convergence, proved in Kalantari [14], ranges from m to the limiting ratio of the generalized Fibonacci numbers of order m.

The following diagram represents the ascending order of convergence of $B_m^{(k)}$, and the corresponding orders for a partial table of iteration functions:

$$
\begin{array}{ccccccccccc}
B_2^{(1)} & \leftarrow & B_2^{(2)} & & & & & & & & \\
\downarrow & & \downarrow & \searrow & & & & & & & \\
B_3^{(1)} & \leftarrow & B_3^{(2)} & \leftarrow & B_3^{(3)} & & & & & & \\
\downarrow & & \downarrow & & \downarrow & \searrow & & & & & \\
B_4^{(1)} & \leftarrow & B_4^{(2)} & \leftarrow & B_4^{(3)} & \leftarrow & B_4^{(4)} & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \searrow & & & \\
B_5^{(1)} & \leftarrow & B_5^{(2)} & \leftarrow & B_5^{(3)} & \leftarrow & B_5^{(4)} & \leftarrow & B_5^{(5)} & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \ldots & & \searrow &
\end{array}
\qquad
\begin{array}{ccccccccccc}
2 & \leftarrow & 1.618 & & & & & & & & \\
\downarrow & & \downarrow & \searrow & & & & & & & \\
3 & \leftarrow & 2.414 & \leftarrow & 1.839 & & & & & & \\
\downarrow & & \downarrow & & \downarrow & \searrow & & & & & \\
4 & \leftarrow & 3.302 & \leftarrow & 2.546 & \leftarrow & 1.927 & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \searrow & & & \\
5 & \leftarrow & 4.236 & \leftarrow & 3.383 & \leftarrow & 2.592 & \leftarrow & 1.965 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \ldots & & \searrow &
\end{array}
$$

The Basic Family is only the first column of the above diagram. The practicality of the multipoints was considered in [22] which gave a computational study of the first nine root-finding methods. These include the Newton, secant, and Halley methods. Our computational results with polynomials of degree up to 30 revealed that for small degree polynomials $B_m^{(k-1)}$ is more efficient than $B_m^{(k)}$, but as the degree increases, $B_m^{(k)}$ becomes more efficient than $B_m^{(k-1)}$. The most efficient of the nine methods is the derivative-free method $B_4^{(4)}$, having a theoretical order of convergence equal to 1.927. Newton's method which is often viewed as the method of choice is in fact the least efficient method. More computational results and a bigger subset of the above infinite table could reveal further practicality of the multipoint iteration family.

The properties of the Basic Family and the Basic Sequence give rise to new algorithmic strategies. When the given input a is close enough to a simple root of the underlying polynomial, for any $m \geq 2$, the Basic Sequence can be turned into an iterative method of order m by replacing the given input a with $B_m(a)$, and repeating. There are many computational implications of these results. As an example, one possible algorithm for polynomial root-finding is the following: for a given input a continue producing $B_m(a)$ until the difference between $B_m(a)$ and $B_{m+1}(a)$ is small. Then for a desirable m replace a with $B_m(a)$ and repeat in order to obtain higher and higher accuracy. It is also possible to switch back and forth between the two schemes.

An algorithm suggested by the global convergence properties of the Basic Sequence in order to compute all the roots of a given polynomial is as follows: first we obtain a rectangle that contains all the roots. Then, by selecting a sparse number of points we generate the corresponding Basic Sequence and thereby obtain a good approximation to a subset of the roots. Then after deflation the same approach can be applied to find additional roots. This method could provide an excellent alternative to the method of Weyl [38] (see also Pan [28] for Weyl's method). Experimentation with this method is intended and will be reported in the future.

In addition to the above multipoint versions of the Basic Family, it is also possible to define an infinite "Truncated Basic Family," as well as a "parameterized Basic Family." Doing polynomiography with these will be the subject of future work. It would also be interesting to consider polynomiography with other families of iteration functions, e.g. the Euler-Schroeder family (see [13], [15]).

Concluding Remarks and the Future of Polynomiography

In this paper I have described some general techniques for creating polynomiographic images. However, even with polynomiography software at one's disposal, from the user's point of view it is essential to have available a very detailed and systematic manual and/or a book. The creation of such publications is one of my future goals. Indeed different users will benefit from different such publications.

It is believed that through various software programs polynomiography could not only grow into a new art form for both professional and non-professional art-making, but also into a tool with enormous applications in the teaching of art and mathematics. One polynomiography software can be used to teach both art and mathematics more effectively; and another can be used for the visualization of polynomial properties by advanced researchers. For instance, I can foresee that some mathematicians will become interested in studying the polynomiography of many important families of polynomials. Polynomiography can be used not only to teach about polynomials and polynomial root-finding, but also about the underlying notions, e.g. complex numbers and operations on them. Effective means of communications of artistic or educational aspects of polynomiography can best be achieved by collaboration with artists, mathematicians, and educators. This is another of my future plans.

Polynomiography also provides a potentially powerful tool for architecture, furniture, and other kinds of design. Some two-dimensional polynomiographs can give rise to three-dimensional objects. These and other three-dimensional applications of polynomiography are yet another one of my future goals.

Finally, I remark that some convergence properties of the Basic Family are extendable to more general analytic functions than polynomials. Thus, analogous to the case of polynomials, it is possible to visualize roots of these more general analytic functions. However, the efficiency of the corresponding visualizations could dramatically decrease. For instance, applying the basic family to a rational function (quotient of two polynomials) is inefficient as we make use of higher order members of the Basic Family. From the visual point of view, the world of polynomials is already infinitely rich. Moreover, polynomials form an important class, which makes them very appealing for conveying many concepts. One of my present goals is to design a new course on the subject of polynomiography with the aim of bringing together students from art and science. While the title of this paper introduces polynomiography as an intersection between mathematics and art, I do believe that it enhances both areas.

References

[1] D. Bini and V. Y. Pan, *Polynomials and Matrix Computations, Vol. 1: Fundamental Algorithms*, Birkhäuser, Boston, Cambridge, MA, 1994.

[2] P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Vol. 161, Springer-Verlag, New York, 1995.

[3] R. L. Devaney, *Introduction to Chaotic Dynamic Systems*, Benjamin Cummings, 1986.

[4] K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley & Sons, 1990.

[5] P. Fatou, Sur les équations fonctionelles, *Bull. Soc. Math. France*, 47 (1919), 161-271.

[6] J. Friedman, On the convergence of Newton's method, *J. of Complexity*, 5 (1989), 12-33.

[7] J. Gerlach, Accelerated convergence in Newton's method, SIAM Review, 36 (1994), 272-276.

[8] J. Gleick, *Chaos: Making a New Science*, Penguin Books, 1988.

[9] E. Halley, A new, exact, and easy method of finding roots of any equations generally, and that without any previous reduction, *Philos. Trans. Roy. Soc. London*, 18 (1694), 136-145.

[10] M.A. Jenkins and J.F. Traub, A three-stage variable-shift iteration for polynomial zeros and its relation to generalized Rayleigh iteration, *Numer. Math.*, 14 (1970), 252-263.

[11] G. Julia, Sur les équations fonctionelles, *J. Math. Pure Appl.*, 4 (1918), 47–245.

[12] B. Kalantari, and I. Kalantari, High order iterative methods for approximating square roots, *BIT* 36 (1996), 395-399.

[13] B. Kalantari, I. Kalantari, and R. Zaare-Nahandi, A basic family of iteration functions for polynomial root finding and its characterizations, *J. of Comp. and Appl. Math.*, 80 (1997), 209-226.

[14] B. Kalantari, On the order of convergence of a determinantal family of root-finding methods, *BIT* 39 (1999), 96-109.

[15] B. Kalantari, Generalization of Taylor's theorem and Newton's method via a new family of determinantal interpolation formulas and its applications, *J. of Comp. and Appl. Math.*, 126 (2000), 287-318.

[16] B. Kalantari, Approximation of polynomial root using a single input and the corresponding derivative values, Technical Report DCS-TR 369, Department of Computer Science, Rutgers University, New Brunswick, New Jersey, 1998.

[17] B. Kalantari, Halley's method is the first member of an infinite family of cubic order root-finding methods, Technical Report DCS-TR 370, Department of Computer Science, Rutgers University, New Brunswick, New Jersey, 1998.

[18] B. Kalantari and J. Gerlach, Newton's method and generation of a determinantal family of iteration functions, *J. of Comp. and Appl. Math.*, 116 (2000), 195-200.

[19] B. Kalantari, New formulas for approximation of π and other transcendental numbers, *Numerical Algorithms* 24 (2000), 59-81.

[20] B. Kalantari, On homogeneous linear recurrence relations and Approximation of Zeros of Complex Polynomials, Technical Report DCS-TR 412, Department of Computer Science, Rutgers University, New Brunswick, New Jersey, 2000, To appear in DIMACS Proceedings on, *Unusual Applications in Number Theory.*

[21] B. Kalantari, An infinite family of iteration functions of order m for every m, Department of Computer Science, Rutgers University, New Brunswick, New Jersey, forthcoming.

[22] B. Kalantari and S. Park, A computational comparison of the first nine members of a determinantal family of root-finding methods, *J. of Comp. and Appl. Math.*, 130 (2001) 197-204.

[23] P. Kirrinnis, Partial fraction decomposition in C(z) and simultaneous Newton iteration for factorization in C[z], *J. of Complexity*, 14 (1998), 378-444.

[24] B. B. Mandelbrot, *The Fractal Geometry of Nature*, New York, W.F. Freeman, 1983.

[25] J. M. McNamee, A bibliography on root of polynomials, *J. of Comp. and Appl. Math.*, 47 (1993), 391-394.

[26] J. W. Neuberger, Continuous Newton's method for polynomials *The Mathematical Intelligencer*, 21 (1999), 18-23.

[27] V. Y. Pan, New techniques for approximating complex polynomial zeros, *Proc. 5th Ann. ACM-SIAM Symp. on Discrete Algorithms*, (1994), 260–270.

[28] V. Y. Pan, Solving a polynomial equation: some history and recent progress, *SIAM Review*, 39 (1997), 187–220.

[29] H. -O. Peitgen and P. H. Richter, *The Beauty of Fractals*, New York, Springer-Verlag, 1992.

[30] H. -O. Peitgen, D. Saupe, H. Jurgens, L. Yunker, *Chaos and Fractals*, New York, Springer-Verlag, 1992.

[31] J. Renegar, On the worst-case complexity of approximating zeros of polynomials, *J. Complexity* 3 (1987), 90-113.

[32] M. Shub, and S. Smale, Computational complexity: On the geometry of polynomials and a theory of cost, Part I, *Ann. Sci. Ecole Norm. Sup.*, 18 (1985), 107-142.

[33] M. Shub, and S. Smale, Computational complexity: On the geometry of polynomials and a theory of cost, Part II, *SIAM J. Comput.* 15 (1986), 145-161.

[34] S. Smale, The fundamental theorem of algebra and complexity theory, *Bull. Amer. Math. Soc.* 4 (1981), 1-36.

[35] S. Smale, Newton's method estimates from data at one point, in *The merging of Disciplines: New Directions in Pure, Applied, and Computational Mathematics*, R.E. Ewing, K.I. Gross, C.F. Martin, eds., 185-196, 1986.

[36] J. F. Traub, *Iterative Methods for the Solution of Equations*, New Jersey, Prentice Hall, 1964.

[37] J. L. Varona, Graphic and numerical comparison between iterative methods, *The Mathematical Intelligencer*, 24 (2002), 37-45.

[38] H. Weyl, Randbemerkungen zu Hauptproblemen der Mathematik. II. Fundamentalsatz der Algebra und Grundlagen der Mathematik, *Math. Z.* 20 (1924), 142–150.

[39] T. J. Ypma, Historical development of Newton-Raphson method, *SIAM Review*, 37 (1995), 531–551.